AN EXPLORATION OF BUCKLING MODES AND DEFLECTION OF A FIXED-GUIDED BEAM

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ABSTRACT

This paper explored the deflection and buckling of fixed-guided beams. It uses an analytical model for predicting the reaction forces, moments, and buckling modes of a fixed-guided beam undergoing large deflections. One of the strengths of the model is its ability to accurately predict buckling behavior and the buckled beam shape. The model for the bending behavior of the beam is found using elliptic integrals. A model for the axial deflection of the buckling beam is also developed based on the equations for stress and strain and the buckling profile of the beam calculated with the elliptic integral solution. These two models are combined to predict the performance of a beam undergoing large deflections including higher order buckling modes. The force vs. displacement predictions of the model are compared to the experimental force vs. deflection data of a bistable mechanism and a thermomechanical in-plane microactuator (TIM). The combined models show good agreement with the force vs. deflection data for each device. The paper’s main contributions include the addition of the axial buckling model to existing beam bending models, the exploration of the deflection domain of a fixed-guided beam, and the demonstration that non-linear finite element models may incorrectly predict a beam’s buckling mode unless unrealistic constraints are placed on the beam.

1 Introduction

Compliant mechanisms are kinematic mechanisms whose motion is derived from the deflection of flexible members rather than rotation of pin joints with rigid links. Compliant mechanisms have been used to replace rigid body mechanisms to reduce their cost and improve their performance due to lower part counts, increased precision, and reduced wear [1]. One example of the effective use of compliant mechanisms is in microelectromechanical systems (MEMS) [2–4].

There are several modeling techniques used to predict the performance of these mechanisms namely, the pseudo rigid body model (PRBM) [1, 5], elastic beam bending solutions [6–8], elliptic integral solutions [1, 9, 10], and finite element models [11–13]. These have provided an increased insight into the behavior of compliant mechanisms by predicting the reaction forces and moments for different arrangements of flexible beams. In several compliant mechanisms, such as compliant bistable mechanisms and thermomechanical microactuators, the motion of the mechanisms relies on the buckling of slender beams [5, 6, 11, 14, 15].

The PRBM accurately predicts the motion of flexible beams but is limited to first mode bending in its accuracy [8]. Non-linear finite element models can predict higher-order modes of buckling, but they have a tendency to predict the wrong buckling mode (third or higher instead of the more realistic second mode seen in practice). To correct this error, an offset force has been applied to bias the buckling of the thin beam [11]. The biasing force corrects the buckling mode but can add an unrealistic
loading condition to the model. In this paper we utilize an elliptic integral model for a beam with fixed-fixed or fixed-guided end condition derived from the general solution in [9]. A similar derivation by Zhou et al. uses a numerical approximation instead of elliptic integrals for the fixed-guided configuration [16]. The model presented here predicts the motion of a fixed-guided beam in both first, second, and higher buckling modes and is capable of accurately predicting which mode corresponds to a given beam end displacement. The novel work in this paper is the insight gained into buckling behavior of fixed-guided beams as a function of end displacement. The behavior over a wide range of deflections is explored, rather than limiting deflection to a single line as been done in the past [16]. This shows the determination of zones of deflection leading to first or second mode bending. This paper concludes with a demonstration of the accuracy of the model by comparing its predictions to the force vs. deflection data of a bistable mechanism and a thermomechanical microactuator.

2 Model

The model developed here to predict the reaction forces and displacements of a flexible beam is made up of two parts. The first is a lateral bending model that accounts for the bending behavior of the beam, and an axial stretching or compression model to account for the forces being transmitted through the beam. The bending model calculates the reaction forces due to bending and these are applied to the axial stretching model to find the displacements due to axial stretching or compression.

2.1 Lateral Bending Model

The bending model developed here presents an analytical solution for the higher order buckling modes for a beam with fixed-guided end conditions. The solution is derived from the equations developed by T. E. Shoup and C. W. McLaran for determining the deflection, reaction forces, and reaction moments for a flexible beam [9]. It assumes a constant second moment of area and modulus of elasticity, and an inextensible beam.

Figure 1 shows the variables used in [9] to calculate the reaction forces and moments of a beam due to bending given any end conditions. The fixed-guided case is modeled as shown in Figure 2 by setting the variables $\alpha_1$ and $\alpha_2$ equal to zero and transforming the reaction forces at the end of the beam using

$$
P = -R\cos\psi \quad (1)
$$

$$
Q = -R\sin\psi \quad (2)
$$

$$
R^2 = P^2 + Q^2 \quad (3)
$$

These equations can then be substituted into the general equations to simplify the bending model. The general equations for displacement due to bending, $b_0$, and $a_0$, corresponding to Figure 1, are

$$
0 = [-a_0(P^2 + Q^2)^{\frac{3}{2}}/\sqrt{EI} + P|E(k, \phi_2) - 2E(k, \phi_1)| - \frac{1}{2}kQ_0(\cos \phi_2 - \cos \phi_1)]
$$

$$
0 = [-b_0(P^2 + Q^2)^{\frac{3}{2}}/\sqrt{EI} + Q|E(k, \phi_2) - 2E(k, \phi_1)| - \frac{1}{2}kP_0(\cos \phi_1 - \cos \phi_2)]
$$

which reduce to

$$
\frac{b_0}{L} = \frac{1}{\sqrt{a}} \{ \sin \psi [2E(\phi_2, k) - 2E(\phi_1, k) - 2F(\phi_2, k) + 2F(\phi_1, k)] + 2k \cos \psi (\cos \phi_1 - \cos \phi_2) \}
$$

FIGURE 1. Diagram of variables used for the general elliptic integral solution.

FIGURE 2. Diagram of variables used for the fixed-guided buckling beam solution.
\[
\frac{a_b}{L} = \frac{-1}{\sqrt{\alpha}} \left\{ \cos \psi \left[ 2E(\phi_2, k) - 2E(\phi_1, k) - 2F(\phi_2, k) + 
2F(\phi_1, k) \right] + 2k \cos \psi (\cos \phi_2 - \cos \phi_1) \right\} 
\] (7)

where the non-dimensional displacements \( \frac{b}{L} \) and \( \frac{w}{L} \) are the vertical and horizontal displacements respectively. \( F(k, \phi) \) is the elliptic integral of the first kind with an amplitude of \( \phi \) and a modulus of \( k \). \( E(k, \phi) \) is the elliptic integral of the second kind, again with an amplitude of \( \phi \) and a modulus of \( k \). \( \phi_1 \) and \( \phi_2 \) are defined below. The magnitude of the reaction force \( R \) is calculated using the general equation

\[
0 = \left[ -L(P^2 + Q^2)^{1/2} / \sqrt{EI} + [F(k, \phi_2) - F(k, \phi_1)] \right] 
\] (8)

which is reduced by substituting from equation (3). The non-dimensional reaction force \( \alpha \) can then be calculated as

\[
\sqrt{\alpha} = L \sqrt{\frac{R}{EI}} = F(k, \phi_2) - F(k, \phi_1) 
\] (9)

The non-dimensional reaction moments at the ends of the beam \( \beta_1 \) and \( \beta_2 \) are defined as

\[
\beta_1 = \frac{M_1 L}{EI} = 2k \sqrt{\alpha} \cos \phi_1 
\] (10)

\[
\beta_2 = \frac{M_2 L}{EI} = 2k \sqrt{\alpha} \cos \phi_2 
\] (11)

where \( \phi_1 \) and \( \phi_2 \) are associated with the origin and displaced end of the beam respectively. For more information on the use and derivation of elliptic integrals, see [17].

The physical significance of the variables \( k \) and \( \phi \) is not readily apparent but the behavior of this system gives some insight into their meaning. For example, \( \phi \) is a variable whose value changes continuously from \( \phi_1 \) at the left end of the beam to \( \phi_2 \) at the right end. The inflection points of the beam occur when \( \phi \) is equal to an odd multiple of \( \frac{\pi}{2} \) such as \( \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2} \), etc. as illustrated in Figure 3. Similarly, larger values of \( k \) roughly correspond to larger beam deflections and forces. For fixed-guided boundary conditions, the variable \( \phi_1 \) can be related to \( \phi_2 \) by the following equation

\[
\sin \phi_1 = \sin \phi_2 
\] (12)

which has an infinite set of solutions for \( \phi_2 \) for any given \( \phi_1 \). \( \phi_2 \) may be found as a function of \( \phi_1 \) by

\[
\phi_2 = n\pi - \phi_1 \quad \text{if } n \text{ is odd} 
\] (13)

\[
\phi_2 = n\pi + \phi_1 \quad \text{if } n \text{ is even} 
\] (14)

where \( n \) is the mode of buckling predicted. Because the inflection points occur at odd multiples of \( \frac{\pi}{2} \), the mode number corresponds to the number of inflection points in the deflected beam shape. Third order, \( (n = 3) \), and higher buckling modes occur in ideal cases but are not encountered in most real systems due to slight asymmetries of the mechanism or due to asymmetric reaction forces and moments. Therefore, first, \( (n = 1) \), and second, \( (n = 2) \), buckling modes are encountered during static bending. The variable \( \phi_1 \) can be calculated using

\[
\sin \phi_1 = \frac{1}{k} \cos \left( \frac{\psi}{2} \right) 
\] (15)

Equations (12)-(15) are specific to the fixed-guided end conditions whereas equations (4)-(11) can be used for a beam with any end conditions.

### 2.2 Axial Stretching Model

In addition to the previous equations for modeling the bending behavior of a fixed-guided beam, a complete model must also incorporate an axial stretching model of the beam to correct for the inextensible assumption in the bending model. The reaction force \( R \) is transmitted through the beam which either compresses or stretches the beam material. This change in length affects the overall displacement of the beam and is therefore added to the displacements calculated with the lateral bending solution above. This approach is accurate as long as the displacements due to axial stretching or compression are small relative to the displacements due to bending. The percentage of displacement due to axial stretching that is considered “small” has not yet been quantified, and is an area we are continuing to study. However, the good agreement between the model and experiments, presented...
below, suggest that this approach is accurate. This model also assumes a constant cross section along the beam’s length.

The vertical displacement due to axial stretching is defined as

\[ b_a = \frac{\alpha}{L^2} \int_0^1 (\cos \phi \cos \theta \sin \theta + \sin \phi \sin^2 \theta) ds' \]  
(16)

and the horizontal displacement is defined as

\[ a_o = \frac{\alpha}{L^2} \int_0^1 (\cos \phi \cos^2 \theta + \sin \phi \sin \theta \cos \theta) ds' \]  
(17)

where the slenderness ratio \( \lambda \) is defined as

\[ \lambda^2 = \frac{AI^2}{L} \]  
(18)

where \( I \) is the second moment of area of the beam cross section as shown on Figure 2. \( ds' \) is a variable transformation defined by the derivative of

\[ s' = s/L \]  
(19)
to non-dimensionalize the displacement equations.

### 2.3 Combined Model

To solve these equations, an iterative solution is needed to solve for \( \phi_1 \) and \( k \) in the bending model above. Either the displacements \( a \) and \( b \) or the reaction force \( R \) and angle \( \psi \) must be given initially. What remain are two equations with two unknowns that can be solved using the Newton Raphson method. These values, and a numerical list of values of \( \phi \) between \( \phi_1 \) and \( \phi_2 \), representing the values of \( \phi \) along the beam, can then be used to find \( \theta \) with

\[ \theta = 2[\sin^{-1}(k \sin \phi) - \sin^{-1}(k \sin \phi_1)] \]  
(20)

After calculating the values of \( \theta \) for the points along the beam, equations (17) and (16) can be numerically integrated to estimate the axial displacement due to stretching or compressing the beam. However, since \( \theta \) is related to \( s \) by \( \phi \) and \( k \), the corresponding values of \( \theta \) for a given set of \( s' \) need to be found to evaluate the function. The values of \( s' \) that correspond to \( \theta \) for a given value of \( \phi \) and \( k \) can be found using

\[ s' = \frac{s}{L} = \frac{1}{\sqrt{\alpha}} (F(k, \phi) - F(k, \phi_1)) \]  
(21)

In summary, the total displacements of the fixed-guided beam are found by adding the axial and the bending displacements from equations (6),(7),(16), and (17). Therefore, the total vertical displacement of the beam is

\[ b = b_a + b_o \]  
(22)

and the total horizontal displacement is

\[ a = a_o + a_a \]  
(23)

In addition to the prediction model, the visual profile of the buckled beams can be found using

\[ \frac{y_i}{L} = -\frac{1}{\sqrt{\alpha}} \left[ \sin \psi \left[ 2E(k, \phi_i) - 2E(k, \phi_1) + F(k, \phi_1) \right] - F(k, \phi_i) \right] + 2k \cos \psi \cos \phi_1 - \cos \phi_i \]  
(24)

\[ \frac{x_i}{L} = -\frac{1}{\sqrt{\alpha}} \left[ \cos \psi \left[ 2E(k, \phi_i) - 2E(k, \phi_1) + F(k, \phi_1) \right] - F(k, \phi_i) \right] + 2k \sin \psi \cos \phi_1 - \cos \phi_i \]  
(25)

where \( x_i \) and \( y_i \) are the coordinates of the profile of the beam corresponding to a value of \( \phi_i \) which ranges from \( \phi_1 \) to \( \phi_2 \). [9]

### 2.4 Computation Methods

This model can be evaluated several different ways. If the horizontal and vertical displacements of the end of the beam are given, \( a \) and \( b \) respectively, the reaction force \( R \) and angle \( \psi \) can be computed. The inverse solution is also possible. If the displacements are given, the values of \( \psi \) and \( k \) can be found using a Newton Raphson iteration technique on equations (6) and (7), after substituting equations (13)-(15) . Once the values of \( k, \phi_1 \), and \( \phi_2 \) are determined, they can be substituted into equation (9) to solve for the reaction force \( R \).

If the reaction force and angle are given, then \( k \) can be solved by iterating on equation (9) after substituting equation (15). \( \phi_1 \) and \( \phi_2 \) can then be found with equations (13)-(15) and then applied to equations (6)-(7) to solve for the displacements. The elliptic integrals are evaluated numerically using the method of the arithmetic-geometric mean and descending Landen transforming defined in [18]. This method gives very fast and accurate evaluation of the elliptic integral functions.

### 3 Results

Figure 4 shows a magnitude and direction vector plot of the reaction forces of the beam for several displacements of the end
FIGURE 4. Vector plot of the reaction force magnitude and direction for an array of beam end displacements. The shaded region corresponds to the displacements that induce second mode buckling and the unshaded corresponds to first mode. Several displaced beam profiles are also shown.

FIGURE 5. Plot showing how the demarcation line between first mode and second mode buckling change with respect to the slenderness ratio.

3.1 Determination of Buckling Mode

An exploration of the beam deflection space reveals that there is a curve in the space which forms the boundary between the first and second (or higher) buckling modes. For beam end deflections above this curve, only first mode solutions are possible. Beam end deflections below the curve require second or higher modes to be used. The boundary curve is formed by choosing the minimum value of $k$ that results in a real result for $\phi_w$ in Equation (15). This will occur when

$$k = \sqrt{\frac{1 + \cos \psi}{2}}$$

(26)

With $k$ determined as a function of $\psi$, the rest of the model can be calculated to determine the beam end deflection on the boundary. Several examples of the boundary curve are shown in Figure 5 for various values of the slenderness ratio and a unit-length beam. Hence, the mode shape capable of reaching a particular beam end deflection is completely determined by the elliptic integral model. In contrast, as we will show, finite element models normally require application of spurious offset loads to correctly predict mode shape.

4 Example Mechanisms

4.1 Bistable Mechanism

The bistable mechanism presented here was designed using the elliptic integral solution, and the measured force vs. displacement is compared to the values predicted by the elliptic integral solution.
4.1 Design In order for the mechanism to be bistable, it must have two stable equilibrium positions and an unstable equilibrium position. This behavior can be seen on the force vs. deflection curve as displacements where the force is equal to zero. Figure 8 shows the stable and unstable positions and the maximum and minimum forces on the force vs. deflection curve of an example bistable mechanism. In addition to the bistability design constraints, the maximum force needed to be less than the maximum capacity of the load cell that would be used during testing and a reasonable safety factor was needed. The elliptic integral model was used to compare possible designs against these constraints. Because of the ease of manufacturing, the mechanism was fabricated out of a single sheet of polypropylene with a Young’s modulus of 1.379 GPa (200 kpsi), and an out of plane height \(h\) of 12.55 mm (0.494 in). The in-plane thickness \(t\) of the thin beams was chosen to be 1.5 mm. A leg length of 70 mm and angle of 5.5° were chosen to increase the maximum force while maintaining the bistability and relatively low stresses of the mechanism. Figure 6 shows the design of the bistable mechanism.

4.1.2 Fabrication The mechanism was manufactured using a CNC mill with a 1/8” end mill, which left a small radius on inside corners between the beams and the frame. Figure 9 shows a photograph of the fabricated mechanism. The measured length of the legs did not include the fillet radius at each end. A
small hole was drilled in the top of the frame to allow a small metal rod to pass through, which would attach the central shuttle to the measuring device. A backing was also attached to the mechanism to prevent flexing in the frame holding the bistable mechanism. This would help ensure the fixed boundary condition by maintaining the end of the legs fixed, as required for the model.

4.1.3 Testing Testing was done in an Instron® tensile testing machine, which applies a given displacement and measures the reaction force of the mechanism. The load cell was calibrated to negate the weight of the mechanism in the experimental data. The bottom of the frame of the bistable mechanism was clamped in the Instron® machine. A metal rod passed through the hole in the frame to attach the mechanism’s shuttle to the load cell. The metal rod is shown in Figure 9. The displacement was applied at a rate of 0.050 mm/s and the reaction force was measured with a 5.8 lb load cell at a sampling frequency of 20 Hz.

4.1.4 Results The measured force vs. displacement curve is compared with the elliptic integral model in Figure 10. The plot shows that the predicted data follows the measured data quite closely. It also compares the predictions of an ANSYS® finite element model with a vertical offset force of 0.025 N applied to the center of the beam in the negative direction to induce second mode buckling. In addition, three finite element models were constructed using beam, plain-strain, and brick elements which, when modeled without the offset force, consistently predicted third mode or higher buckling. If the loading was performed
with the smallest possible displacement steps starting with the offset force applied and removing it on a later step, the results were the same. As soon as the offset force was removed, the analysis would again predict third mode or higher buckling. The offset force was minimized to minimize the error due to the offset while maintaining second mode buckling.

The force vs. displacement curve in Figure 10 is a good example of the relationship between the buckling mode and the reaction force of the beams. Figure 11 shows the first mode buckling of the beams with one inflection point which approximately corresponds to the non-linear curves at the beginning and end of the force-deflection behavior of the bistable mechanism. The linear portion of the curve in the middle of Figure 10 approximately corresponds to the second mode buckling behavior shown in Figure 12.

Table 1 compares the measured values for the maximum and minimum forces, their locations, and the locations of the unstable and stable equilibrium positions to the predicted values from the elliptical integral model. The discrepancies between the measured and predicted values may have been caused by stress relaxation in the polypropylene [19, 20]. Another factor may be the non-ideal boundary conditions, due to the fillets at the ends of the legs or possible flexing of the frame.

4.2 Thermomechanical In-Plane Microactuator

4.2.1 Mechanism Description This second mechanism example is a microelectromechanical system or MEMS microactuator designed and tested by J. Wittwer [11]. This thermomechanical in-plane microactuator (TIM) was fabricated using a surface micromachining process (SUMMIT®) which vapor deposits polysilicon in layers onto a nitride film on a silicon wafer. These layers are separated by a layer of silicon oxide which is then removed by an etching process leaving polysilicon structures which, depending on the design, can move and conduct electricity. The TIM designed by Wittwer is constructed out of three layers of polysilicon in a chevron shape as shown in Figure 13. As current passes through the thin beams on the TIM, shown as i on the figure, their temperature increases causing them to expand. With the opposed arrangement of the beams, this expansion causes the beams to buckle to the side, which pushes the center shuttle upwards. Once the legs of the TIM reach a steady state temperature, they can be modeled as fixed-guided beams with an increased length. The motion of the TIM is shown as the dotted line in Figure 13.

4.2.2 Results For this mechanism it is important to know the force vs. displacement along the displacement path of the shuttle to understand how this actuator will function when it interacts with the system being actuated. Each force vs. displacement data set on the plot was taken with a different heating current. For more detail on the experimental setup and measurement techniques, see [11]. In order to correctly model the thermal expansion of the beams, a finite difference method was used to estimate the steady state temperature of any point along the beam [21]. The total linear thermal expansion of the beam was

$$\Delta L = \int_0^L \alpha_L(T(s))(T(s) - T_0)ds$$  \hspace{1cm} (27)$$

where the temperature profile $T(s)$ was calculated with the heat transfer model. $\alpha_L(T)$ is the temperature dependent linear thermal expansion coefficient of polysilicon and $T_0$ is the initial ambient temperature. Figure 14 shows the increased length due to the thermal expansion $\Delta L$. The dashed lines on the figure show the buckling modes of the beam at the beginning and end of the displacement path. This buckling is caused by the opposed arrangement of the beams in the TIM. Though the beam’s length increases, the symmetry of the TIM prevents the beam from expanding straight out; instead, it must expand by moving along the dotted line shown in the figure.
FIGURE 15. Comparison of the force vs. displacement data for a thermomechanical in-plane microactuator and the model predictions. The solid lines represent the elliptic integral solution for each heating current.

Figure 15 shows that the elliptical integral model is able to adequately predict the second mode buckling region of the force vs. displacement curve. The second mode buckling region corresponds to the linear portion of data on the left side of the 13.3 mA and 15 mA data set. However, at the 13.3 mA and 15 mA current levels, the model predicts a higher force than is measured. A hypothesis for this difference is that as the temperature of the polysilicon increases, its Young’s modulus decreases [12]. The model presented here accounts for the temperature dependence of the thermal expansion coefficient, but it assumes that the modulus of elasticity is constant. As the modulus decreases with an increased temperature, the force predictions of the 13.3 mA and 15 mA cases would proportionally decrease and the prediction would become more accurate.

Finite element models were also able to predict second mode buckling for the TIM but they required an offset force applied to the center of the beam to induce second mode buckling, as with the bistable mechanism. Without the offset the finite element model produced a third mode buckling solution, which only applies to the ideal case of perfect symmetry [11, 22], and an inaccurate force vs. deflection curve similar to the third mode prediction curves in Figure 10 with the bistable mechanism.

5 Conclusions and Recommendations

The model developed here presents a method to predict buckling behavior of fixed-guided beams. It combines an elliptic integral model for the buckling of a flexible beam with a numerical axial deflection solution. Together these models provide a method for calculating the reaction force of a beam for a given displacement, or calculating the deflection of a beam given a directional force. In particular, this model is capable of accurately modeling the beam in both the first and second mode buckling regions with a capability of modeling the higher modes if desired. The model was used to explore a wide deflection space for a fixed-guided beam. The results showed different zones of deflection for first and second mode bending. The zones are clearly separated using the model.

Following the development of the model, its predictions were compared to two compliant mechanisms whose motions require the beams to buckle in both first and second mode. There is adequate agreement between the model and the measured force vs. displacement behavior of the mechanisms. In both cases the second mode buckling region showed a linear force vs. deflection relationship that was modeled well with the analytical model. In addition, non-linear finite element models were shown to pro-
duce an incorrect solution unless an unrealistic transverse load was placed at the center of the beam. To further improve the solution for the bistable mechanism, a correlation to account for stress relaxation of the materials could be developed. Also, in the case of the TIM, the temperature dependence of the Young’s modulus could be incorporated. However, despite these minor adjustments, the model follows the behavior of real systems and provides insight into the motion of fixed-guided beams.

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